

BI-POISSON PROCESS

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ABSTRACT. We study a two parameter family of processes with linear regressions and linear conditional variances. We give conditions for the unique solution of this problem, and point out the connection between the resulting Markov processes and the generalized convolutions introduced by Bożejko and Speicher [2].

1. INTRODUCTION

Throughout this paper $(X_t)_{t \geq 0}$ is a square integrable stochastic process such that for all $t, s \geq 0$

$$(1) \quad E(X_t) = 0, \quad E(X_t X_s) = \min\{t, s\}.$$

Consider the σ -fields $\mathcal{G}_{s,u} = \sigma\{X_t : t \in [0, s] \cup [u, \infty)\}$, $\mathcal{F}_s = \sigma\{X_t : t \in [0, s]\}$, $\mathcal{G}_u = \sigma\{X_t : t \in [u, \infty)\}$. We assume that the process has linear regressions,

Assumption 1. For all $0 \leq s < t < u$,

$$(2) \quad E(X_t | \mathcal{G}_{s,u}) = \mathbf{a}X_s + \mathbf{b}X_u,$$

where

$$(3) \quad \mathbf{a} = \mathbf{a}(t|s, u) = \frac{u-t}{u-s}, \quad \mathbf{b} = \mathbf{b}(t|s, u) = \frac{t-s}{u-s}.$$

are the deterministic functions of $0 \leq s < t < u$.

We also assume that the process has quadratic conditional variances;

$$(4) \quad E(X_t^2 | \mathcal{G}_{s,u}) = \mathbf{A}X_s^2 + \mathbf{B}X_s X_u + \mathbf{C}X_u^2 + \mathbf{D} + \alpha X_s + \beta X_u,$$

where $\mathbf{A} = \mathbf{A}(t|s, u)$, $\mathbf{B} = \mathbf{B}(t|s, u)$, $\mathbf{C} = \mathbf{C}(t|s, u)$, $\mathbf{D} = \mathbf{D}(t|s, u)$, $\alpha = \alpha(t|s, u)$, $\beta = \beta(t|s, u)$ are the deterministic functions of $0 < s < t < u$.

Generically, conditions (1), (2), and (4) imply that there are five real parameters $q, \eta, \theta, \sigma, \tau$ such that

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$$(5) \quad \mathbf{A}(t|s, u) = \frac{(u-t)(u(1+\sigma t) + \tau - qt)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(6) \quad \mathbf{B}(t|s, u) = \frac{(u-t)(t-s)(1+q)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(7) \quad \mathbf{C}(t|s, u) = \frac{(t-s)(t(1+\sigma s) + \tau - qs)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(8) \quad \mathbf{D}(t|s, u) = \frac{(u-t)(t-s)}{u(1+\sigma s) + \tau - qs},$$

$$(9) \quad \alpha(t|s, u) = \frac{(u-t)(t-s)}{u(1+\sigma s) + \tau - qs} \times \frac{u\eta - \theta}{u-s},$$

$$(10) \quad \beta(t|s, u) = \frac{(u-t)(t-s)}{u(1+\sigma s) + \tau - qs} \times \frac{\theta - s\eta}{u-s}.$$

This gives after a calculation

$$(11) \quad \begin{aligned} \text{Var}(X_t|\mathcal{G}_{s,u}) = & \frac{(u-t)(t-s)}{u(1+\sigma s) + \tau - qs} \left(1 + \sigma \frac{(uX_s - sX_u)^2}{(u-s)^2} + \eta \frac{uX_s - sX_u}{u-s} \right. \\ & \left. + \tau \frac{(X_u - X_s)^2}{(u-s)^2} + \theta \frac{X_u - X_s}{u-s} + (1-q) \frac{(X_u - X_s)(sX_u - uX_s)}{(u-s)^2} \right), \end{aligned}$$

compare [7, Proposition 2.5]. (Recall that the conditional variance of X with respect to a σ -field \mathcal{F} is defined as $\text{Var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - (E(X|\mathcal{F}))^2$.)

In [7] we prove that the solution of equations (1), (2), (11) exists and is unique when $-1 < q \leq 1$, and $\sigma = \eta = 0$; it is then given by the Markov process which we called q -Meixner process. (The case $q = 1$ yields Lévy processes, and was studied earlier by several authors, see [13], and the references therein.) Due to the invariance of this problem under the symmetry that maps (X_t) to the process $(tX_{1/t})$, processes that satisfy (11) with $-1 < q \leq 1$, $\tau = \theta = 0$ are also Markov, and can be expressed in terms of the q -Meixner processes as $tX_{1/t}$. The main feature of these examples are trivial (constant) conditional variances in one direction of time, which leads to technical simplifications.

The study of the remaining cases poses difficulties, as several steps from [7] break down. In this paper we consider the next simplest case, which one may call the free bi-Poisson processes. The q -Poisson processes, in particular, the classical Poisson process and the free Poisson process, have linear conditional variances when conditioned with respect to the future, and constant conditional variances when conditioned with respect to the past. The bi-Poisson process has linear conditional variances under each uni-directional conditioning; it corresponds to the choice of $\sigma = \tau = 0$ in (11). The adjective "free" refers to $q = 0$. The role of these simplifying conditions seems technical: linear conditional variances imply that all moments are finite, see Lemma 3.2; additional condition that $q = 0$ allows us to guess useful algebraic identities between the orthogonal polynomials in Proposition 2.2. These considerations lead to the following.

Assumption 2. For all $0 \leq s < t < u$,

$$(12) \quad \begin{aligned} \text{Var}(X_t|\mathcal{G}_{s,u}) = & \frac{(u-t)(t-s)}{u} \left(1 + \eta \frac{uX_s - sX_u}{u-s} + \theta \frac{X_u - X_s}{u-s} + \frac{(X_u - X_s)(sX_u - uX_s)}{(u-s)^2} \right). \end{aligned}$$

In Section 2 we construct the Markov process with covariances (1), linear regressions (2), and conditional variances (12) for a large set of real parameters η, θ . In Section 3 we show that the solution is unique. In Section 4 we point out that when $\theta = 1$ the one-dimensional distributions of the bi-Poisson process are closed under a generalized free convolution.

2. EXISTENCE

If $\eta = 0$, formula (12) coincides with [7, (28)] with $\tau = q = 0$, so the corresponding Markov process exists and is determined uniquely, see [7, Theorem 3.5]. Since the transformation $X_t \mapsto tX_{1/t}$ switches the roles of η, θ , the case $\theta = 0$ follows, too. We may therefore restrict our attention to the case $\eta\theta \neq 0$. The construction of the processes is based on the idea already exploited in [7]; namely, we construct the transition probabilities of the suitable Markov process, by defining the corresponding orthogonal polynomials. Under current assumptions, this task requires more work as we need to ensure that the coefficient at the third term of the recurrence for the polynomials is non-negative. The construction relies on new identities between the orthogonal polynomials, which are used to verify the martingale polynomial property (24); the latter property fails for more general values of parameters in (11).

2.1. One dimensional distributions. We begin by carefully examining the "candidate" for the one dimensional distribution of X_t . For $t > 0$, let $p_0(x; t) = 1$, and consider the following monic polynomials $\{p_n(x; t) : n \geq 1\}$ in variable x .

$$(13) \quad xp_0 = p_1 + 0p_0,$$

$$(14) \quad xp_1 = p_2 + (t\eta + \theta)p_1 + tp_0,$$

$$(15) \quad xp_n = p_{n+1} + (t\eta + \theta)p_n + t(1 + \eta\theta)p_{n-1}, \quad n \geq 2.$$

From the general theory of orthogonal polynomials, if $1 + \eta\theta \geq 0$ then there exists a unique probability measure π_t such that $p_n(x; t)$ are orthogonal with respect to π_t , see [8]. We will need the following.

Lemma 2.1.

$$(16) \quad \pi_t(\{x : 1 + \eta x < 0\}) = 0.$$

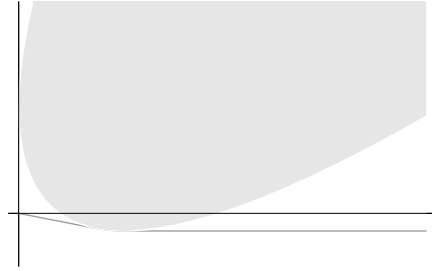


FIGURE 1. The vertical cross-section of the gray area is the support of π_t for $t > 0$; the gray lines represent the support of the discrete part. This picture represents the case $\eta > 0, \theta > 0$.

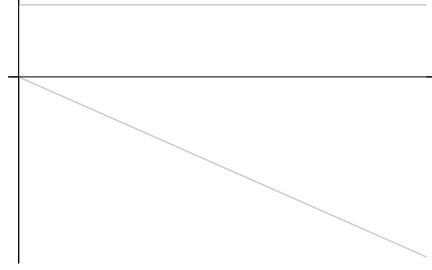


FIGURE 2. The support of the distribution of X_t for $t > 0$ in the degenerate case $\eta < 0, \theta > 0, 1 + \eta\theta = 0$.

Proof of Proposition 2.1. There is nothing to prove when $\eta = 0$, so without loss of generality we assume that $\eta \neq 0$.

If $1 + \eta\theta = 0$ then the recurrence is degenerate and the distribution is supported at zeros of polynomial $p_2(x) = x^2 - (t\eta + \theta)x - t$; this follows from the fact that all higher order polynomials are multiples of p_2 . The support $\text{supp}(\pi_t) = \{-t/\theta, -1/\eta\}$, see Fig. 2, is disjoint with the open set $\{x : 1 + \eta x < 0\}$, ending the proof in this case.

If $1 + \eta\theta > 0$, then (15) is a constant coefficient recurrence which has been analyzed by several authors, see [11]. The Cauchy transform

$$G(z) = \int \frac{1}{z - x} \pi_t(dx)$$

is given by the corresponding continued fraction,

$$G(z) = \frac{1}{z - \frac{t}{z - (t\eta + \theta) - \frac{t(1 + \eta\theta)}{z - (t\eta + \theta) - \frac{t(1 + \eta\theta)}{\ddots}}}}$$

which after a calculation gives

$$(17) \quad G(z) = \frac{z(1 + 2\eta\theta) + t\eta + \theta - \sqrt{(z - (t\eta + \theta))^2 - 4t(1 + \eta\theta)}}{2(1 + z\eta)(t + z\theta)}.$$

The Stieltjes inversion formula gives the distribution π_t as the limit in distribution as $\varepsilon \rightarrow 0^+$ of the absolutely continuous measures $-\frac{1}{\pi} \Im G(x + i\varepsilon)dx$. This gives

$$(18) \quad \pi_t(dx) = \frac{t}{2\pi} \frac{\sqrt{4t(1 + \eta\theta) - (x - t\eta - \theta)^2}}{(x\eta + 1)(x\theta + t)} 1_{(x - t\eta - \theta)^2 < 4t(1 + \eta\theta)} + p(t)\delta_{-t/\theta} + q(t)\delta_{-1/\eta}.$$

The weights at the discrete point masses are

$$p(t) = \frac{-((1 + \eta\theta)t - \theta^2)/\theta + \varepsilon|(1 + \eta\theta)t - \theta^2|/|\theta|}{2(\theta - \eta t)}$$

and

$$q(t) = \frac{\eta(t - (1 + \eta\theta)/\eta^2) + \varepsilon|\eta||t - (1 + \eta\theta)/\eta^2|}{2(\eta t - \theta)},$$

where the sign $\varepsilon = \varepsilon(t, \eta, \theta) = \pm 1$ is selected simultaneously for both expressions by the appropriate choice of the branch of the square root. We found that a practical

way to choose the sign is to select $\varepsilon = \pm 1$ so that both expressions give a number in the interval $[0, 1]$; in our setting this determines ε uniquely for every choice of parameters, after all the cases are considered.

It is easy to check that the support of the absolutely continuous part of π_t does not intersect the set $\{x : 1 + x\eta < 0\}$. The support of the discrete part consists of at most two-points: $\{-t/\theta, 1/\eta\}$. Thus the only possibility for the set $\{x : 1 + x\eta < 0\}$ to carry positive π_t -probability is when $-t/\theta \in \{x : 1 + x\eta < 0\}$. This is possible only if $\eta\theta > 0$ and t is large enough. The Stieltjes inversion formula gives the weight of $-t/\theta$ as

$$p(t) = \frac{(\theta^2 - (1 + \eta\theta)t)_+}{\theta^2 - t\eta\theta}.$$

Thus the point $-t/\theta$ carries positive probability $p(t)$ only for $t < \frac{\theta^2}{1+\eta\theta} \leq \theta/\eta$; on the other hand, $-t/\theta \in \{x : 1 + x\eta < 0\}$ only for $t > \theta/\eta$. \square

2.2. Transition probabilities. Fix $0 < s < t$, and let $x \in \mathbb{R}$ be such that $1 + x\eta \geq 0$. We define monic polynomials in variable y by the three-step recurrence

$$Q_0(y; t, x, s) = 1,$$

$$Q_1(y; t, x, s) = y - x,$$

$$yQ_1(y; t, x, s) = Q_2(y; t, x, s) + ((t-s)\eta + \theta)Q_1(y; t, x, s) + (t-s)(1+x\eta)Q_0(y; t, x, s),$$

and for $n \geq 2$ by the constant coefficients recurrence

(19)

$$yQ_n(y; t, x, s) = Q_{n+1}(y; t, x, s) + (t\eta + \theta)Q_n(y; t, x, s) + t(1 + \eta\theta)Q_{n-1}(y; t, x, s).$$

We define $P_{s,t}(x, dy)$ as the (unique) probability measure which makes the polynomials $\{Q_n(y; t, x, s) : n \in \mathbb{N}\}$ orthogonal; this is possible whenever $1 + \eta\theta \geq 0$ and $1 + x\eta \geq 0$, a condition that is satisfied if X_s has the distribution $\pi_s(dy) = P_{0,t}(0, dy)$, see (16). Since the coefficients of the three step recurrence (19) are bounded, it is well known that measures $P_{s,t}(x, dy)$ have bounded support.

The next step is to prove that $P_{s,t}(x, dy)$ form a consistent family of measures, so that they indeed define the transition probabilities of the Markov chain which starts at the origin. To this end, we need the following algebraic relations between the polynomials. These relations are a more complicated version of [6, Theorem 1] and [7, Lemma 3.1].

Proposition 2.2. *For $n \geq 0$*

$$(20) \quad Q_n(z; x, u, s) = Q_n(y; x, t, s) + \sum_{k=0}^{n-1} B_k(y; x, t, s) Q_{n-k}(z; y, u, t),$$

where $B_0 = 1$ and

$$B_1(y; x, t, s) = Q_1(y; x, t, s) - (t-s)\eta B_0,$$

$$(21) \quad B_k(y; x, t, s) = Q_k(y; x, t, s) - t\eta B_{k-1}(y; x, t, s), \quad k = 2, 3, \dots$$

Additionally, for $n \geq 1$

$$(22) \quad Q_n(y; x, t, s) = \sum_{k=0}^n \tilde{B}_{n-k}(x; s) (p_k(y; t) - p_k(x; s)),$$

where $\tilde{B}_k(x; s) = B_k(0; x, 0, s)$ are linear (affine) functions in variable x .

Proof. Let

$$\phi(\zeta; y, x, t, s) = \sum_{n=0}^{\infty} \zeta^n Q_n(y; t, x, s)$$

be the generating function of Q_n . Since $\phi(\zeta; y, x, t, s) = 1 + z \sum_{n=0}^{\infty} \zeta^n Q_{n+1}(y; t, x, s)$, a calculation based on recurrence (19) shows that

$$\phi(\zeta; y, x, t, s) = \frac{1 + \zeta(t\eta + \theta - x) + \zeta^2(s + sy\eta - tx\eta + t\eta\theta)}{1 + \zeta(t\eta + \theta - y) + \zeta^2 t(1 + \eta\theta)}.$$

From (21) we get a similar expression for the generating function of B_n . Namely,

$$\psi(\zeta; y, x, t, s) = \sum_{n=0}^{\infty} \zeta^n B_n(y|x, t, s) = \frac{\phi(\zeta; y, x, t, s) + \eta s \zeta}{1 + \eta t \zeta}.$$

This gives

$$\psi(\zeta; y, x, t, s) = \frac{1 + \zeta(s\eta + \theta - x)z + s(1 + \eta\theta)\zeta^2}{1 + \zeta(t\eta + \theta - y) + t(1 + \eta\theta)\zeta^2}.$$

It is now easy to verify that the two generating functions are connected by

$$(23) \quad \phi(\zeta; z, x, u, s) - \phi(\zeta; y, x, t, s) = \psi(\zeta; y, x, t, s)(\phi(\zeta; z, y, u, t) - 1),$$

which implies (20). Since $\psi(\zeta; y, x, t, s)\psi(\zeta; x, y, s, t) = 1$ from (23) we get

$$\phi(\zeta; z, y, u, t) = 1 + \psi(\zeta; x, y, s, t)(\phi(\zeta; z, x, u, s) - \phi(\zeta; y, x, t, s)).$$

Since $p_n(x, t) = Q_n(x; 0, t, 0)$ setting $x = 0, s = 0$ proves (22). \square

We now follow the argument from [7, Proposition 3.2] and verify that probability measures $P_{s,t}(x, dy)$ are the transition probabilities of a Markov process.

Proposition 2.3. *If $0 \leq s < t < u$ and $1 + \eta\theta \geq 0$, then*

$$P_{s,u}(x, \cdot) = \int P_{t,u}(y, \cdot) P_{s,t}(x, dy).$$

Proof. Let $\nu(A) = \int P_{t,u}(y, A) P_{s,t}(x, dy)$. To show that $\nu(dz) = P_{s,u}(x, dz)$, we verify that the polynomials $Q_n(z; x, u, s)$ are orthogonal with respect to $\nu(dz)$. Polynomials Q_n satisfy the three-step recurrence (19); it suffices therefore to show that for $n \geq 1$ these polynomials integrate to zero. Since $\int Q_n(z; y, u, t) P_{t,u}(y, dz) = 0$ for $k \geq 1$, by (20) we have

$$\begin{aligned} \int Q_n(z; x, u, s) \nu(dz) &= \int Q_n(y|x, t, s) P_{s,t}(x, dy) \\ &+ \sum_{k=0}^{n-1} \int B_k(y; x, t, s) \left(\int Q_{n-k}(z; y, u, t) P_{t,u}(y, dz) \right) P_{s,t}(x, dy) = 0. \end{aligned}$$

\square

For $1 + \eta\theta \geq 0$, let (X_t) be the Markov process with the transition probabilities $P_{s,t}(x, dy)$, $X_0 = 0$.

Lemma 2.4. *For $t > s, n \in \mathbb{N}$ we have*

$$(24) \quad E(p_n(X_t; t) | \mathcal{F}_s) = p_n(X_s; s).$$

Proof. By definition, for $n \geq 1$ we have $E(Q_n(X_t; X_s, t, s)|X_s) = 0$. Since $p_1(x, t) = x$, and $Q_1(y; x, t, s) = y - x$, by the Markov property (24) holds true for $n = 1$.

Suppose that (24) holds true for all $n \leq N$. Then (22) implies

$$0 = E(Q_{N+1}(X_t; X_s, t, s)|X_s) = \tilde{B}_0(X_s; s) (E(p_{N+1}(X_t; t)|X_s) - p_{N+1}(X_s; s)).$$

Since $\tilde{B}_0 = 1$, this proves that $E(p_{N+1}(X_t; t)|X_s) = p_{N+1}(X_s; s)$, which by the Markov property implies (24) for $N + 1$. \square

Theorem 2.5. *Suppose $1 + \eta\theta \geq 0$ and (X_t) is the Markov process with transition probabilities $P_{s,t}(x, dy)$, and $X_0 = 0$. Then (1), (2), and (12) hold true.*

Proof. Condition (1) holds true as $E(X_t) = \int p_1(x; t)p_0(x; t)\pi_t(dx) = 0$, and for $s < t$ from (24) we get $E(X_s X_t) = E(X_s E(p_1(X_t; t)|\mathcal{F}_s)) = \int p_1^2(x; s)\pi_s(dx) = \int (p_2(x; s) + (s\eta + \theta)p_1(x; s) + s)\pi_s(dx) = s$.

Since X_t are bounded, polynomials are dense in $L_2(X_s, X_u)$. Thus by the Markov property to prove (2) we only need to verify that

$$(25) \quad \begin{aligned} & E(p_n(X_s; s)X_t p_m(X_u; u)) \\ &= \mathbf{a}(t|s, u)E(X_s p_n(X_s; s)p_m(X_u; u)) + \mathbf{b}(t|s, u)E(p_n(X_s; s)X_u p_m(X_u; u)) \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $0 < s < t$.

For the proof of (12), we need to verify that for any $n, m \geq 1$ and $0 < s < t$

$$(26) \quad \begin{aligned} & E(p_n(X_s, s)X_t^2 p_m(X_u, u)) \\ &= \mathbf{A}E(X_s^2 p_n(X_s, s)p_m(X_u, u)) + \mathbf{B}E(X_s p_n(X_s, s)X_u p_m(X_u, u)) \\ & \quad + \mathbf{C}E(p_n(X_s, s)X_u^2 p_m(X_u, u)) + \alpha E(X_s p_n(X_s, s)p_m(X_u, u)) \\ & \quad + \beta E(p_n(X_s, s)X_u p_m(X_u, u)) + \mathbf{D}E(p_n(X_s, s)p_m(X_u, u)), \end{aligned}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \alpha, \beta$ are given by equations (5), (6), (7), (8), (9), (10):

$$\begin{aligned} \mathbf{A} &= \frac{u-t}{u-s}, \mathbf{B} = \frac{(t-s)(u-t)}{(u-s)u}, \mathbf{C} = \frac{(t-s)t}{(u-s)u}, \mathbf{D} = \frac{(t-s)(u-t)}{u}, \\ \alpha &= \frac{(t-s)(u-t)(u\eta - \theta)}{(u-s)u}, \beta = \frac{(t-s)(t-u)(s\eta - \theta)}{(u-s)u} \end{aligned}$$

It is convenient to introduce the notation Ep_m^2 for $E(p_m^2(X_s; s))$. Recall that (15) implies $Ep_1^2 = s$, and for $n \geq 1$

$$(27) \quad Ep_{n+1}^2 = s(1 + \eta\theta)Ep_n^2,$$

see [8, page 19].

An efficient way to verify (25) and (26) is to use generating functions. For $s \leq u$, let

$$\phi_0(z_1, z_2, s) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s; s)p_m(X_u; u)).$$

From (24) it follows that $\phi_0(z_1, z_2, s)$ does not depend on u , and from (27) it follows that

$$\phi_0(z_1, z_2, s) = \frac{1 - z_1 z_2 \eta \theta s}{1 - z_1 z_2 s(1 + \eta \theta)}.$$

Consider now the generating function

$$\phi_1(z_1, z_2, s, t) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s; s) X_t p_m(X_u; u)).$$

From (24) and (15) we get

$$\begin{aligned} \phi_1(z_1, z_2, s, t) &= \sum_{n=0}^{\infty} z_1^n E \left(p_n(X_s; s) \left(X_t + z_2 X_t p_1(X_t; t) + \sum_{m=2}^{\infty} z_2^m X_t p_m(X_t; t) \right) \right) \\ &= \sum_{n=0}^{\infty} z_1^n E \left(p_n \left(p_1 + z_2(p_2 + (t\eta + \theta)p_1 + tp_0) + \sum_{m=2}^{\infty} z_2^m (p_{m+1} + (t\eta + \theta)p_m + t(1 + \eta\theta)p_{m-1}) \right) \right). \end{aligned}$$

Thus

$$\phi_1(z_1, z_2, s, t) = \left(\frac{1}{z_2} + t\eta + \theta \right) (\phi_0(z_1, z_2, s) - 1) + z_2 t(1 + \eta\theta) \phi_0(z_1, z_2, s) - \eta\theta t z_2,$$

which gives

$$\phi_1(z_1, z_2, s, t) = \frac{sz_1 + tz_2 + sz_1 z_2(t\eta + \theta)}{1 - sz_1 z_2(1 + \eta\theta)}.$$

Since a calculation verifies that

$$\phi_1(z_1, z_2, s, t) = \mathbf{a}(t|s, u) \phi_1(z_1, z_2, s, s) + \mathbf{b}(t|s, u) \phi_1(z_1, z_2, s, u),$$

(see (3)) from this (25) follows. Finally, for $s \leq t_1 \leq t_2 \leq u$ consider the generating function

$$\phi_2(z_1, z_2, s, t_1, t_2) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s; s) X_{t_1} X_{t_2} p_m(X_u; u)).$$

Another calculation based on (24) and (15) gives

$$\begin{aligned} &\phi_2(z_1, z_2, s, t_1, t_2) \\ &= \left(\frac{1}{z_2} + t_2\eta + \theta \right) (\phi_1(z_1, z_2, s, t_1) - \phi_1(z_1, 0, s, t_1)) + z_2 t_2(1 + \eta\theta) \phi_1(z_1, z_2, s, t_1) - z_1 z_2 s \eta \theta. \end{aligned}$$

A computer assisted calculation now verifies that

$$\begin{aligned} &\phi_2(z_1, z_2, s, t, t) \\ &= \mathbf{A} \phi_2(z_1, z_2, s, s, s) + \mathbf{B} \phi_2(z_1, z_2, s, s, u) + \mathbf{C} \phi_2(z_1, z_2, s, u, u) + \mathbf{D} \phi_0(z_1, z_2, s) \\ &\quad + \alpha \phi_1(z_1, z_2, s, s) + \beta \phi_1(z_1, z_2, s, u), \end{aligned}$$

which proves (26). \square

3. UNIQUENESS

We first state the main result of this section.

Theorem 3.1. *Suppose $(X_t)_{t \geq 0}$ is a centered square-integrable separable stochastic process with covariance (1). If (X_t) satisfies (2) and (12) with $1 + \eta\theta \geq 0$, then X_t is the Markov process, as defined in Theorem 2.5.*

The proof of Theorem 3.1 is based on the method of moments.

Lemma 3.2. *Under the assumptions of Theorem 3.1 $E(|X_t|^p) < \infty$ for all $p > 0$.*

Proof. This result follows from [4, Corollary 4]. To use this result, fix $t_1 < t_2$ and let $\xi_1 = t_1^{-1/2} X_{t_1}$, $\xi_2 = t_2^{-1/2} X_{t_2}$. Then their correlation $\rho = E(\xi_1 \xi_2) = \sqrt{t_1/t_2} \in (0, 1)$. It remains to notice that $E(\xi_i|\xi_j) = \rho \xi_j$ and the variances $\text{Var}(\xi_i|\xi_j) = 1 - \rho^2 + a_j \xi_j$; these relations follow from taking the limits $s \rightarrow 0$ or $u \rightarrow \infty$ in (2) and (12). Thus by [4, Corollary 4], $E(|\xi_1|^p) < \infty$ for all $p > 0$. \square

The next result is closely related to [5, Proposition 3.1] and [13, Theorem 2].

Lemma 3.3. *Suppose X_t has covariance (1), and satisfies conditions (2) and (12). If $0 \leq s < t$ then $E(X_t^k|\mathcal{F}_s)$ is a monic polynomial of degree k in variable X_s with uniquely determined coefficients.*

Proof. By Lemma 3.2, $E(|X_t^n|) < \infty$ for all n . Clearly, $E(X_t^k|\mathcal{F}_s)$ is a unique monic polynomial of degree k when $k = 0, 1$. Suppose that the conclusion holds true for all $s < t$ and all $k \leq n$ for some integer $n \geq 1$. Multiplying (2) by X_u^n and applying to both sides conditional expectation $E(\cdot|\mathcal{F}_s)$, we get

$$E(X_t E(X_u^n|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{a} X_s E(X_u^n|\mathcal{F}_s) + \mathbf{b} E(X_u^{n+1}|\mathcal{F}_t).$$

Using the induction assumption, we can write this equation as

$$(28) \quad E(X_t^{n+1}|\mathcal{F}_s) = \mathbf{a} X_s^{n+1} + \mathbf{b} E(X_u^{n+1}|\mathcal{F}_t) + f_n(X_s),$$

where f_n is a unique polynomial of degree at most n .

Multiplying (4) by X_u^{n-1} and applying $E(\cdot|\mathcal{F}_s)$ to both sides, we get

$$E(X_t^2 E(X_u^{n-1}|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{A} X_s^2 E(X_u^{n-1}|\mathcal{F}_s) + \mathbf{B} X_s E(X_u^n|\mathcal{F}_t) + \mathbf{C} E(X_u^{n+1}|\mathcal{F}_t) + \dots$$

Using the induction assumption, we can write this equation as

$$(29) \quad E(X_t^{n+1}|\mathcal{F}_s) = (\mathbf{A} + \mathbf{B}) X_s^{n+1} + \mathbf{C} E(X_u^{n+1}|\mathcal{F}_t) + g_n(X_s),$$

where g_n is a unique polynomial of degree at most n . Since $\mathbf{b} - \mathbf{C} \neq 0$, subtracting (28) from (29) we get

$$E(X_u^{n+1}|\mathcal{F}_t) = \frac{\mathbf{a} - \mathbf{A} - \mathbf{B}}{\mathbf{C} - \mathbf{b}} X_s^{n+1} + h_n(X_s),$$

where h_n is a (unique) polynomial of degree at most n .

From (5), (6), (7) we get

$$\frac{\mathbf{a} - \mathbf{A} - \mathbf{B}}{\mathbf{C} - \mathbf{b}} = \frac{1 + \sigma u}{1 + \sigma s} = 1,$$

as $\sigma = 0$. Thus $E(X_t^{n+1}|\mathcal{F}_s) = X_s^{n+1} + h_n(X_s)$ is a monic polynomial of degree $n + 1$ in variable X_s with uniquely determined coefficients. \square

Proof of Theorem 3.1. Denote by (Y_t) the Markov process from Theorem 2.5. Recall that Y_t are bounded random variables for any $t > 0$. We will show that by the method of moments that (X_t) and (Y_t) have the same finite dimensional distributions.

By Theorem 2.5, process (Y_t) satisfies the assumptions of Lemma 3.3. Therefore, for $n \geq 0$

$$(30) \quad E(Y_t^n|\mathcal{F}_s) = Y_s^n + h_{n-1}(Y_s),$$

$$(31) \quad E(X_t^n|\mathcal{F}_s) = X_s^n + h_{n-1}(X_s)$$

with the same polynomial h_{n-1} . From this, we use induction to deduce that all mixed moments are equal. Taking $s = 0$, from (30) and (31) we see that $E(X_t^n) =$

$E(Y_t^n)$ for all $n \in \mathbb{N}, t > 0$. Suppose that for some $k \geq 1$ and all $0 < t_1 < t_2 < \dots < t_k$, all $n_1, \dots, n_k \in \mathbb{N}$ we have

$$E(X_{t_1}^{n_1} X_{t_2}^{n_2} \dots X_{t_k}^{n_k}) = E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \dots Y_{t_k}^{n_k}).$$

Then from (30) and (31), by the induction assumption we get for any $t > t_k$ and $n \in \mathbb{N}$

$$\begin{aligned} E(X_{t_1}^{n_1} X_{t_2}^{n_2} \dots X_{t_k}^{n_k} X_t^n) &= E(X_{t_1}^{n_1} X_{t_2}^{n_2} \dots X_{t_k}^{n_k} E(X_t^n | \mathcal{F}_{t_k})) \\ &= E\left(X_{t_1}^{n_1} X_{t_2}^{n_2} \dots X_{t_{k-1}}^{n_{k-1}} X_{t_k}^{n_k} (X_t^n + h_{n-1}(X_{t_k}))\right) \\ &= E\left(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \dots Y_{t_{k-1}}^{n_{k-1}} Y_{t_k}^{n_k} (Y_t^n + h_{n-1}(Y_{t_k}))\right) \\ &= E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \dots Y_{t_k}^{n_k} E(Y_t^n | \mathcal{F}_{t_k})) = E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \dots Y_{t_k}^{n_k} Y_t^n). \end{aligned}$$

Since $t > t_k$ and $n \in \mathbb{N}$ are arbitrary, this shows that all mixed moments of the $k+1$ -dimensional distributions match. \square

Corollary 3.4. *Suppose (X_t) is a Markov process from Theorem 2.5 with parameters $\eta = \theta$. Then the process $(tX_{1/t})_{t>0}$ has the same finite dimensional distributions as process $(X_t)_{t>0}$.*

Proof. It is well known that (1), and hence (2), are preserved by the transformation $(X_t) \mapsto (tX_{1/t})$. A calculation shows that if $\eta = \theta$ then the conditional variance (12) is also preserved by this transformation. Thus by Theorem 3.1, both processes have the same distribution. \square

Remark 3.1. With more work and suitable additional assumptions, Theorem 2.5 and Theorem 3.1 can perhaps be extended to conditional variances (11) with $\tau \neq 0$ as long as $\sigma = 0, q = 0$. Generalizations to $-1 < q < 1$, are hampered by the lack of suitable identities for the corresponding orthogonal polynomials. When $\sigma \neq 0$, an additional difficulty arises from the fact that the martingale polynomial property (24) fails.

4. GENERALIZED CONVOLUTIONS

Let $\tilde{\pi}_t$ be the measure determined by polynomials (15) with $\eta = 0$. Then $\tilde{\pi}_t$ is a univariate distribution of the Markov process Y_t from [7, Theorem 3.5] with $\tau = q = 0$. Since this is a classical version of the free centered Poisson process, it is known that $\tilde{\pi}_t$ form a semigroup with respect to the free-convolution, $\tilde{\pi}_{t+s} = \tilde{\pi}_t \boxplus \pi_s$.

It is somewhat surprising that there is a generalization of the convolution that works in a more general case; this generalization, the c -convolution, is defined in [2] and studied in [1], [3], [9], [10].

For our purposes the most convenient definition of the c -convolution is analytic approach from [1, Theorem 5.2]. According to this result, the c -convolution $(\mu_1, \nu_1) \star_c (\mu_2, \nu_2)$ is a binary operation on the pairs of probability measures (μ_j, ν_j) , defined as follows. Let g_j, G_j be the Cauchy transforms

$$g_j(z) = \int \frac{1}{z-x} \mu_j(dx), \quad G_j(z) = \int \frac{1}{z-x} \nu_j(dx)$$

On the first component of a pair, the generalized convolution acts just via the free convolution. Let $k_j(z)$ be the inverse function of $g_j(z)$ in a neighborhood of ∞ , and define $r_j(z) = k_j(z) - 1/z$. The free convolution μ of measures μ_1, μ_2 is defined as

the unique probability measure with the Cauchy transform $g(z)$ which solves the equation

$$g(z) = \frac{1}{z - r_1(g(z)) - r_2(g(z))},$$

see [12].

To define the second component of the c -convolution, let

$$R_j(z) = k_j(z) - 1/G_j(k_j(z)).$$

The second component of the c -convolution is defined as the unique probability measure ν with the Cauchy transform

$$G(z) = \frac{1}{z - R_1(g(z)) - R_2(g(z))}.$$

We write

$$(\mu, \nu) = (\mu_1, \nu_1) \star_c (\mu_2, \nu_2);$$

thus we require that the pair of functions (r, R) as defined above be additive with respect to the c -convolution. Functions r, R are the so called r/R -transforms and define the c -free cumulants, which have interesting combinatorial interpretation.

Denote by $\mathcal{L}(X)$ the distribution of a random variable X . Let Y_t be the free Poisson process, i.e. the Markov process from Theorem 2.5 with parameter $\eta = 0, \theta \in \mathbb{R}$. Let X_t be the Markov process from Theorem 2.5 with parameters $\eta, \theta \in \mathbb{R}, 1 + \eta\theta \geq 0$.

Proposition 4.1. *If $\theta = 1$, then pairs of measures $(\mathcal{L}(Y_t + t(1 + \eta)), \mathcal{L}(X_t + t))$ form a semigroup with respect to the c -convolution,*

$$\begin{aligned} & (\mathcal{L}(Y_{t+s} + (t+s)(1 + \eta)), \mathcal{L}(X_{t+s} + t + s)) \\ &= (\mathcal{L}(Y_t + t(1 + \eta)), \mathcal{L}(X_t + t)) \star_c (\mathcal{L}(Y_s + s(1 + \eta)), \mathcal{L}(X_s + s)). \end{aligned}$$

Proof. A calculation shows that $r_t(z) = \frac{t(1+\eta)}{1-z}$. Since $r_{t+s}(z) = r_t(z) + r_s(z)$, this verifies that indeed measures $\mathcal{L}(Y_t + t(1 + \eta))$ form a semigroup with respect to free convolution.

Another calculation shows that

$$R_t(z) = \frac{t}{1-z}.$$

Since $R_{t+s}(z) = R_t(z) + R_s(z)$, this verifies the c -convolution property for the second component. \square

Measures π_t for $\theta = 1$ occur also in the Poisson Limit theorem for c -convolutions; the Cauchy transform derived in [1, page 380] up to centering is equivalent to (17). The conversion is accomplished by shifting argument in (17) and making in the resulting expression $G(z - t)$ one of the following substitutions

$$\{\theta \rightarrow 1, \eta \rightarrow \frac{-\alpha + \beta}{\alpha}, t \rightarrow \alpha\},$$

$$\{\theta \rightarrow -\alpha + \beta, \eta \rightarrow \frac{1}{\alpha}, t \rightarrow \alpha\}.$$

(The second substitution is equivalent to the first one applied to the time-reversal $tX_{1/t}$ of the bi-Poisson process.)

Remark 4.1. After the first draft of this paper was written, we learned about another version of the generalized convolution, the t -convolution from [10]; this convolution acts on single probability measures rather than on pairs, and could have been used in Proposition 4.1 instead of the c -convolution. (The case $\theta \neq 1$ still poses a challenge.)

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